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DEFORMATION OF A DROP IN A VISCOUS STREAM AND CONDITIONS OF EXISTENCE OF ITS EQUILIBRIUM FORM*

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The difference method is used for obtaining a solution of the problem of unsteady motion of a drop in a stream, taking into account its deformation under conditions of axial symmetry. The fluid inside and outside the drop is assumed viscous and incompressible. The stable forms of drop are represented for various Reynolds and Weber numbers of external stream. By analyzing the conditions for normal stresses at the drop boundary, the critical Weber number was obtained, which establishes the conditions of existence of equilibrium form of the drop.

1. In spherical system of coordinates that are attached to the axisymmetric drop so that the coordinate origin is at the center of mass of the drop and the polar axis (the axis of symmetry) is directed along the stream the system of equations that define the flow inside and outside the drop is of the form

$$\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}-\frac{v_\theta^2}{r}=\frac{2}{\mathrm{Re}_k}\left\{\Delta v_r-\frac{2}{r^2}\left(v_r+\frac{\partial(v_\theta\sin\theta)}{\sin\theta\,\partial\theta}\right)\right\}-\varkappa_k\frac{\partial P}{\partial\theta} \tag{1.1}$$

$$\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{v_r v_\theta}{r}=\frac{2}{\mathrm{Re}_k}\left\{\Delta v_\theta+\frac{2}{r^2}\left(\frac{\partial v_r}{\partial\theta}-\frac{v_\theta}{2\sin^2\theta}\right)\right\}-\frac{\varkappa_k}{r}\frac{\partial P}{\partial\theta}$$

$$\Delta P=\frac{2}{r\varkappa_k}\left\{\frac{\partial(v_r,v_\theta)}{\partial(r,\theta)}+v_\theta\left(\frac{\partial r_v}{\partial r}-\frac{v_\theta}{r}\operatorname{ctg}^2\theta\right)-v_r\left(\frac{v_r}{r}\frac{\partial v_r}{\partial r}\frac{2v_\theta\operatorname{ctg}\theta}{r}\right)\right\}$$

$$\left(\mathrm{Re}_k=\frac{2R_0u_0\rho_k}{\mu_k},\quad \varkappa_k=\begin{cases}\rho_2/\rho_1, & k=1\\ 1, & k=2\end{cases},\quad P=p+\frac{\omega}{\varkappa_k}r\cos\theta\right)$$

where the last equation is the corollary of the first two and the condition of incompressibility. In the equations r is the dimensionless coordinate expressed in units of radius R_0 equivalent to the volume of the spherical drop, θ is the polar angle $(0\leqslant\theta\leqslant\pi)$, v_r, v_θ are velocity components normalized with respect to velocity u_0 of the stream at infinity at initial instant of time, subscript $k=1$ and $k=2$ relate, respectively, to the inner and outer regions, P is the modified pressure normalized with respect to $\rho_2u_0^2$, and ω is the drop acceleration in the stream normalized with respect to u_0^2/R_0.

As the drop boundary are satisfied the conditions of continuity of the normal and tangent components of velocity vector the continuity of the tangent stress, the equality of the normal stress discontinuity to the capillary pressure, and the kinematic condition

$$[v_s]=0,\ [v_n]=0,\ [\mathbf{s}\Pi\mathbf{n}]=0,\quad [\mathbf{n}\Pi\mathbf{n}]=\frac{2}{\mathrm{We}^{-1}}\left(\frac{1}{R_1}+\frac{1}{R_2}\right) \tag{1.2}$$

$$\mathrm{We}=2\rho_2u_0^2R_0/\gamma,\ dr/dt=v_r,\ d\theta/dt=v_\theta/r$$

where $\mathbf{s},\mathbf{n}$ are unit vectors tangential and normal to the drop boundary ($\mathbf{n}$ is directed inside the external stream), Π is the stress tensor, γ is the surface tension coefficient, and R_1, R_2 are the principal curvature radii of the drop boundaries normalized with respect to the radius of the spherical drop.

Away from the drop the velocity components are subject to conditions

$$r\to\infty,\quad v_r=\cos\theta\left(1-\int_0^t\omega dt\right),\quad v_\theta=-\sin\theta\left(1-\int_0^t\omega dt\right) \tag{1.3}$$

At the axis of symmetry the condition of axial symmetry must be satisfied. At the initial instant of time the drop form and the vector of velocity must be specified. The latter

*Prikl.Matem.Mekhan.,Vol.46,No.6,pp.1045-1049,1982 must satisfy the equation of continuity, the first three conditions of (1,2) and condition (1.3) for the equivalence of solution of the system of Eqs.(1.1) to solution of the initial system of Navier–Stokes equations, and the equation of continuity require the fulfillment of the equation of continuity at the boundaries.

In the local Cartesian coordinates system the conditions of continuity of the tangent stress together with the condition of incompressibility can be written in a form /1/ that will prove to be more convenient for subsequent difference approximation. In the course of solving the problem, the alteration of drop boundaries necessitates coordinate transformation of the form

$$\theta=\theta,\quad \eta=r\left(\frac{k_1}{R_3}-\left(\frac{k_1}{R_3}-\frac{1}{\delta}\right)\frac{R_3-r}{R_3-\delta}\right)$$

where k_1, R_3 are constants, and $\delta(\theta, t)$ is the function defining the boundaries of the drop. In new coordinates the drop boundary is a circle of unit radius.

The conditions of continuity of tangential stress together with the condition incompressibility and the equality of the normal stress discontinuity to the capillary pressure in new coordinates are written as follows:

$$\left[\mu\left(\left(a^{\pm}\eta_r'+\frac{a^{\mp}\eta_\delta'\delta_\theta'}{\delta}\right)\frac{\partial Q^{\pm}}{\partial\eta}+\frac{a^{\mp}}{\delta}\frac{\partial Q^{\pm}}{\partial\theta}+\frac{v_n\sin\varphi+v_s\cos\varphi}{\delta\sin\theta}\right)\right]=0 \tag{1.4}$$

$$\left[-P+\frac{4}{\varkappa\,\mathrm{Re}}\left(\left(\eta_r'\cos\alpha-\frac{\eta_\delta'\delta_\theta'}{\delta}\sin\alpha\right)\frac{\partial v_n}{\partial\eta}-\frac{\sin\alpha}{\delta}\frac{\partial v_n}{\partial\theta}\right)\right]=$$

$$\frac{2}{\mathrm{We}}\left(\frac{1}{R_1}+\frac{1}{R_2}-\mathrm{Bo}\,(1-\varkappa_1)\,\delta\cos\theta\right);\quad \mathrm{Bo}=\frac{\omega\rho_1R_0^2}{\gamma}$$

$$Q^{\pm}=v_s\pm v_n,\ a^{\pm}=\cos\alpha\pm\sin\alpha,\ \alpha=\theta-\varphi$$

where φ is the angle between the normal to the drop boundary and the axis of symmetry.

2. The problem was solved by the difference method. For solving the first two equations of motion (1.1) written in new coordinates, the scheme of variable directions was used. The equation for pressure was solved by the method of establishments, based on the scheme of variable directions. The writing of conditions of tangent stress together with the condition of incompressibility in the form (1.4) enabled the use of the scheme of running calculations for the determination of velocity vector components along the drop boundary /1/.

The general order of calculation consists of the following.

The solution of equation for pressure is obtained. A difference analog of the first of Eqs.(1.1) is used for the determination of the pressure at the drop boundary by the external stream. The pressure on the drop boundary the internal stream is obtained the difference analog of the second of Eqs.(1.4).

The velocity vector components are determined for the inside as well as for the outside stream. This is obtained by additional iterations for compliance with the conditions of continuity at the drop boundary.

Acceleration of the drop in the stream is determined, the velocity vector components are recalculated away from the drop, and from the difference analog of the kinematic condition the drop form is established.

The described method was used for solving the problem of deformation of the orignally spherical drop placed in the stream that moves at infinity at velocity u_0.. The dimensionless parameters that define the process were selected in such a way that the characteristic time of drop formation and the characteristic time of damping its oscillations due to dissipation is considerably smaller than the characteristic time of variation of difference of velocities between the external stream and the center of mass of the drop. In all calculations the time of variation of difference of velocities during the time of drop deformation to the maximum value did not exceed 7% of velocity at the initial instant of time.

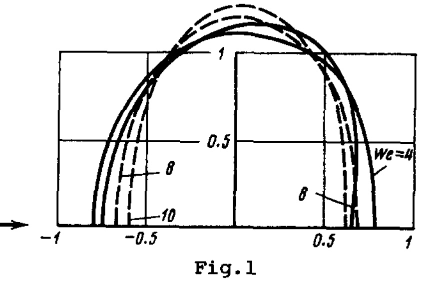

Fig.1

The shape of deformed drop in the stream is shown in Fig.1 for $\mathrm{Re}_2 = 10$, $\mu_1/\mu_2 = 10$, $\rho_1/\rho_2 = 10$ (solid lines) and for $\mathrm{Re}_2 = 40$, $\mu_1/\mu_2 = 40$, $\rho_1/\rho_2 = 10$ (dash lines). It is seen that the degree of deformation of the drop at the same values of the Weber number depends on the Reynolds number of the outer flow. For the dependence of the middle cross section of the deformed drop to the initial radius of the spherical drop on the Weber number is of the form

$$R'/R_0 = 1 + 0.027\,\mathrm{We}$$

On the basis of experimental investigation at $\mathrm{Re}_2 = 100-700$ /2/ an analogous dependence with a coefficient 0.03 at the Weber number was proposed in /3/.

3. It was not possible to establish the drop form for $\mathrm{We} > 20$. We shall show that this may be due to the impossibility of existence of a steady form of drop for such Weber numbers. For simplicity, let us first consider the case in which it is possible to disregard the inertia and viscous forces of motion inside the drop in comparison with the inertia forces of the oncoming stream. The mentioned conditions are satisfied if

$$\frac{\rho_1 u_1^2}{\rho_2 u_2^2} \sim \frac{\rho_1 \mu_2^2}{\rho_2 \mu_1^2} \mathrm{Re}_2 \ll 1, \quad \frac{\mu_1 u_1}{2R_0 \rho_2 u_2^2} \sim \frac{1}{\sqrt{\mathrm{Re}_2}} \ll 1 \tag{3.1}$$

In the considered case inside the drop only the hydrostatic forces produced by the acceleration of the drop in the stream are essential.

Since (3.1) implies that $\mathrm{Re}_2 \gg 1$, in the equation of discontinuity of normal stresses we shall disregard the viscous component of normal stress in the external stream in comparison with the pressure. We locate the origin of cylindrical coordinates r and z at the stern point of the drop and direct the z axis against the stream. The equation of discontinuity of normal stresses then is written in the form

$$\mp \frac{1}{r} \frac{d}{dz} \frac{r}{\sqrt{1 + r_z'^2}} - (1 - \varkappa_1)\,\mathrm{Bo}\, z + c = \frac{\mathrm{We}}{2} P_2, \quad c = \frac{\mathrm{We}}{2} P_{10} \tag{3.2}$$

where P_{10} is the pressure at the drop boundary at $z = 0$, and the signs plus and minus relate to the forward and rear fronts of the drop. If pressure $P_2(r)$ is known, Eq. (3.2) together with additional conditions of symmetry and the expression for the dimensionless volume

$$r_z'|_{z=0,\, r=0} = r_z'|_{z=z_0,\, r=0} = 0, \quad V = \frac{4}{3}\pi \tag{3.3}$$

enable us to determine the drop steady form.

The necessary condition of solvability of problem (3.2), (3.3) is of the following form:

$$-(1 - \varkappa_1)\,\mathrm{Bo}\, \frac{4}{3}\pi = \frac{\mathrm{We}}{2} \int_0^{r_1} (P_+ - P_-)\, r\, dr \tag{3.4}$$

where r_1 is the dimensionless radius of the middle cross section of the drop, and the subscripts plus and minus denote the pressure at its forward and rear of its ends. At high Re_2 numbers the acceleration and the Bo number are determined only by the pressure drop arising in flow separation at the drop, and (3.4) reduces to the equation of motion of the center of mass of the drop. Hence, if the acceleration appearing in the Bo number is found from the equation of motion, (3.4) is satisfies.

Let for some values of parameters of the problem function $P_2(r)$ and the steady form of the drop satisfying (3.2) and (3.3) be known. Let us add to numbers Bo, We and function $P_2(r)$ small increments $\delta\,\mathrm{Bo}$, $\delta\,\mathrm{We}$ and $\delta P_2(r)$. It is required to determine the new equilibrium surface close to the initial one. For this we linearize problem (3.2), (3.3). We assume that each point $\mathbf{X}$ has received a small increment $\delta\mathbf{X}$ which we expand in a sum of vectors $\delta_1\mathbf{X} = \mathbf{n}N$ perpendicular and $\delta_2\mathbf{X}$ tangent to the surface.

We specify the initial surface in parametric form $r(s)$ and $z(s)$, selecting as the parameter s the distance of the considered point to one of the poles measured along the meridian arc. Then for the perturbation of the surface $N(s)$ we obtain the following problem:

$$LN - \delta c_1 r = \frac{d}{ds}[rN_s'] + \left[r\left(r_{ss}''^2 + z_{ss}''^2\right] + \frac{r_s'^2}{r} + \mathrm{Bo}\,(1 - \varkappa_1)\, r r_s'\right] N - \delta c_1 r = \tag{3.5}$$

$$-\left\{\delta\left(\frac{\mathrm{We}}{2} P_2\right) + z\delta\,[(1 - \varkappa_1)\,\mathrm{Bo}]\right\} r; \quad \delta c_1 = \delta c + \mathrm{Bo}\,(1 - \varkappa_1)\,\delta r(0)$$

$$N_s'|_{s=0} = N_s'|_{s=s_0} = 0, \quad \int_0^{s_0} Nr\, ds = 0 \tag{3.6}$$

where s_0 is the distance along the meridian between poles.

A problems similar to (3.2), (3.3) and (3.5), (3.6) but with the flow in the drop taken into consideration, were used in /4,5/ for determining the drop form of not too high Numbers We and Bo

Besides problem (3.5), (3.6) we shall consider the corresponding to it homogeneous boundary value problem. Let us assume that the homogeneous problem for some selection of flow parameters appearing not only explicitly in it, but also implicitly in the form of the drop, has the nontrivial solution $\bar{N}(s)$. By virtue of self-adjointness of the homogeneous and the inhomogeneous boundary value problem, it follows from the Green's formula that the inhomogeneous problem has a solution only then, when the condition of orthogonality of solution $\bar{N}(s)$, and the right-hand side of (3.5) is satisfied.

The solution of the homogeneous problem $\bar{N}(s)$ depends only on form of the drop and the number $(1-\varkappa_1)\mathrm{Bo}$ for the imput unperturbed values of parameters. Unlike $\bar{N}(s)$, $\delta(^1/_2\,\mathrm{We}P_2)+z\delta[(1-\varkappa_1)\,\mathrm{Bo}]$ depends not only on the drop form but on the perturbations of the stream at the external boundary. Such perturbations may be, for example, the velocity variation (including the inhomogeneous), density of the stream variation at infinity, etc. Hence we can expect a situation will arise in which the orthogonality condition of solution $\bar{N}(s)$ to the right-hand side of (3.5) is not satisfied and, consequently, the solution of the perturbed problem (3.5), (3.6) will not exist. Moreover even when it is satisfied, steady random variation in the external flow may lead to its breakdown.

However the nonexistence of solution of problem (3.5), (3.6) does not imply the nonexistence of solution in the input nonlinear statement. Besides, the conditions of existence and stability of solution of problem (1.1)—(1.3) may be impaired prior to reaching parameters for which the solution of problem (3.5), (3.6) does not exist. Hence the answer to the question what takes place in fact in the "suspisious" situation, has to be looked for in the comparison with the numerical calculations and the experiment.

Note that similar approach can be used for determining the critical totality of parameters when the drop form is affected by internal motion and the viscosity in the drop.

The "suspisious" parameters were determined by a numerical solution of problem (3.5), (3.6) in /6/.

The first part of this paper was concerned with the investigation of the case when $\mathrm{Re}_2=40$ and the perturbations that have arisen due to increase of the Weber number. It appeared that in this case the homogeneous problem corresponding to problem (3.5), (3.6) has a nontrivial solution at $\mathrm{We}_*=20.6$. The drop form was approximated by an ellipsoid of revolution, the ratio of whose semiaxes was extrapolated by the results of calculations. The "suspisious" totality of parameters was calculated also, for the conditions of experiments described in /2/. The values of $\mathrm{We}_*=21$ obtained for these conditions was in satisfactory agreement with those of experimentally obtained in /2/.

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